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# Lazy plumes

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We examine the dynamics of turbulent lazy plumes rising from horizontal area sources and from vertically distributed line sources into a quiescent environment of uniform density. First, we consider plumes with internal buoyancy flux gain and, secondly, plumes from horizontal area sources that have significant momentum flux deficits. We re-cast the conservation equations of Morton *et al.* (1956) for a constant entrainment coefficient ($\alpha$) in terms of three dimensionless parameters: the plume radius $\beta$; a parameter $\Gamma$ characterizing the local balance of momentum, buoyancy and volume fluxes; and a parameter $\Lambda$ that characterizes the rate of internal buoyancy flux gain with height. For a plume with a linear internal buoyancy flux gain with height the flow is shown to be a constant-velocity lazy plume. For highly lazy area sources we derive exact solutions for the key plume parameters in terms of $\Gamma$ and an approximate solution for the variation of $\Gamma$ with height. We show that near the source there is a region of zero entrainment.

## 1. Introduction

The entrainment assumption, and flux conservation equations for a turbulent plume in a quiescent environment were introduced by Morton, Taylor & Turner (1956, hereafter referred to as MTT). They assumed a constant entrainment coefficient $\alpha$ and developed equations for the local fluxes of volume, momentum and buoyancy in an axisymmetric Boussinesq turbulent plume. This work was extended by Morton (1959) who considered general sources including those with non-zero values of the source volume flux ($Q_0$), momentum flux ($M_0$) and buoyancy flux ($F_0$) and, in particular, plumes with an excess of momentum flux at the source – so-called ‘forced plumes’. Morton & Middleton (1973) further extended this work by developing scale diagrams indicating the location of particular points of interest in a plume (e.g. the neck height, virtual origin location, etc.) as a function of the source parameter

$$\Gamma(z=0)=\Gamma_0=\frac{5Q_0^2F_0}{4\alpha M_0^{5/2}}, \tag{1.1}$$

defined at the source $z=0$. This work identified both ‘forced plumes’ and ‘lazy plumes’ which were classified based on the balance of fluxes (1.1) at the source. A lazy plume arises from a source with a deficit of momentum flux compared to a pure plume with the same source buoyancy and volume fluxes. Lazy plumes have also been referred to as ‘distributed’ plumes (Caulfield & Woods 1995). However, we retain the term ‘lazy’ as it implies reduced forcing, whereas ‘distributed’ implies an area source which could be lazy ($\Gamma > 1$), pure ($\Gamma = 1$) or forced ($\Gamma < 1$).

The MTT plume conservation equations have also been used to solve for the virtual origin location of a general plume source, e.g. Morton (1959), Caulfield &

Woods (1995) and Hunt & Kaye (2001), the latter validating the theoretical predictions of the origin location by means of measurements of saline plumes in the laboratory. Further extensions to the MTT plume theory have been used to solve a range of problems involving convection from localized sources including the behaviour of turbulent fountains (Bloomfield & Kerr 2000) and non-Boussinesq plumes (Rooney & Linden 1996; Carlotti & Hunt 2005). More recently, Fannelop & Webber (2003) have examined large area source plumes with the non-Boussinesq terms included.

The bulk of the work on plumes has assumed that the entrainment coefficient $\alpha$, which relates the radial entrainment velocity to the mean vertical velocity in the plume, is independent of height (MTT). The appropriateness of this assumption, however, has been called into question by a number of authors. For example, List & Imberger (1973) argue that the entrainment coefficient for a jet $\alpha_j = 0.056$ is considerably lower than that for a pure plume $\alpha_p = 0.085$ (see Turner 1986 for a detailed discussion of the entrainment assumption) and that using a constant $\alpha$ $(= \alpha_p)$ to model forced plumes will overpredict the radial growth rate. They suggest an entrainment function with $\alpha$ varying with the square of the local plume Richardson number $R(z) \sim QF^{1/2}M^{-5/4}$ (see also Kotsovinos & List 1977). More recently Kaminski, Tait & Carazzo (2005) argued that as well as the Richardson number dependence, $\alpha$ is a function of the rate of change in the relative radii of the velocity and buoyancy profiles, a so-called self-similarity drift. All these Richardson-number-dependent models potentially lead to large values of $\alpha$ for large-Richardson-number (i.e. lazy) flows.

Another turbulent flow where the constant-$\alpha$ hypothesis has been called into question is a plume with internal buoyancy flux gain. Bhat & Narasimha (1996) argue that the traditional MTT conservation equations overpredict the volume flux in clouds where latent heat release leads to an increase in the plume's buoyancy flux away from its source. This observation was the motivation behind the vortex dynamics model of Sreenivas & Prasad (2000) that predicts a *reduced* entrainment coefficient when 'off-source' heating occurs and when plumes are accelerating.

In this paper we focus on lazy plumes in uniform quiescent surroundings, and re-write the MTT equations for a constant entrainment coefficient in terms of the dimensionless plume parameter $\Gamma$, a dimensionless radius $\beta$ and a 'heating' parameter $\Lambda$ (§ 2). We then examine steady solutions of these equations for both constant buoyancy flux plumes and plumes with internal buoyancy flux gains (§ 3). Approximate solutions for the near-source flow are presented for plumes with a constant buoyancy flux and large source $\Gamma_0$, and these show that, to leading order, there is zero entrainment near the source (§ 4). Conclusions are drawn in § 5.

## 2. Plume conservation equations

We start with the MTT conservation equations for a constant-buoyancy-flux plume from a localized horizontal source in a quiescent uniform environment. The equations are written for Gaussian profiles in terms of the fluxes of volume ($\pi Q$), momentum ($\pi M/2$) and buoyancy ($\pi F/2$):

$$\frac{\mathrm{d}Q}{\mathrm{d}z} = 2\alpha M^{1/2}, \quad \frac{\mathrm{d}M}{\mathrm{d}z} = 2\frac{QF}{M}, \quad \frac{\mathrm{d}F}{\mathrm{d}z} = 0. \tag{2.1}$$

For a general area (or distributed) source, the source conditions are taken to be

$$Q = Q_0, \quad M = M_0, \quad F = F_0 \quad \text{at} \quad z = 0, \tag{2.2}$$

where the subscript 0 signifies quantities at the actual source $z = 0$. From the source conditions two length scales can be established, namely the source radius length $L_Q(0)$ and the source momentum jet length $L_M(0)$ written as

$$L_Q(0) = \frac{5Q_0}{6\alpha M_0^{1/2}}, \quad L_M(0) = \left(\frac{5M_0^{3/2}}{9\alpha F_0}\right)^{1/2}, \tag{2.3}$$

respectively. The length scale $L_Q$ *is* the distance from the actual source to the virtual source of a pure plume while the jet length is the distance over which the source momentum flux plays a significant role in the dynamics. It is also possible to define an acceleration length scale $L_A \sim Q^{3/5}F^{-1/5} \sim L_Q^{3/5}L_M^{2/5}$ that is the lazy plume analogue of the forced plume jet length. However, we elect to use $L_Q$ rather than $L_A$ as it is the natural length scale that appears when non-dimensionalizing the MTT equations (see Hunt & Kaye 2001), and because the source parameter $\Gamma_0$ used in the literature is the square of the ratio of the length scales $L_Q$ and $L_M$:

$$\Gamma_0 = \frac{L_Q(0)^2}{L_M(0)^2} = \frac{5Q_0^2F_0}{4\alpha M_0^{5/2}}, \tag{2.4}$$

where $\Gamma_0 = 0$ for a pure jet and $\Gamma_0 = 1$ for a pure plume. Although $\Gamma$ has been used to characterize the source fluxes, it, along with $L_Q$ and $L_M$, can be evaluated at any height

$$\Gamma(z) = \frac{5Q(z)^2F(z)}{4\alpha M(z)^{5/2}}. \tag{2.5}$$

We now consider the simplest case of a plume with a linear increase in buoyancy flux with height ($\epsilon$). This is the case considered by Bhat & Narasimha (1996) and will allow qualitative comparison with their results. For a more general variable-buoyancy-flux model see Caulfield & Woods (1995). The conservation of buoyancy flux (2.1) now becomes

$$\frac{\mathrm{d}F}{\mathrm{d}z} = \epsilon. \tag{2.6}$$

This new term introduces an additional source length scale $L_F(0)$ given by

$$L_F(0) = \frac{F_0}{\epsilon}, \tag{2.7}$$

and we can define a second dimensionless parameter $\Lambda$ as

$$\Lambda(z) = \frac{L_Q^2(z)}{L_F^2(z)} = \frac{25Q(z)^2\epsilon^2}{36\alpha^2 M(z)F(z)^2}. \tag{2.8}$$

$\Lambda$ provides a measure of the off-source buoyancy flux gain relative to the local plume fluxes. We now introduce the non-dimensional fluxes and height scaling

$$m = \frac{M}{M_0}, \quad q = \frac{Q}{Q_0}, \quad f = \frac{F}{F_0}, \quad \zeta = \frac{z}{L_Q(0)}. \tag{2.9}$$

The radius $b(z)$ and vertical velocity $W(z)$ are scaled on their source values such that

$$\beta = \frac{b}{b_0} = \frac{q}{m^{1/2}}, \quad w = \frac{W}{W_0} = \frac{m}{q}. \tag{2.10}$$

Note the radius scaling is not $b/L_Q(0)$ but $5b/6\alpha L_Q(0)$. Equations (2.1) now become

$$\frac{\mathrm{d}q}{\mathrm{d}\zeta} = \frac{5}{3}m^{1/2}, \quad \frac{\mathrm{d}m}{\mathrm{d}\zeta} = \frac{4}{3}\Gamma_0\frac{qf}{m}, \quad \frac{\mathrm{d}f}{\mathrm{d}\zeta} = \Lambda^{1/2}. \tag{2.11}$$

The local parameters $\Gamma$ and $\Lambda$ may be expressed, relative to their source values, as

$$\frac{\Gamma}{\Gamma_0} = \frac{q^2 f}{m^{5/2}}, \quad \frac{\Lambda}{\Lambda_0} = \frac{q^2}{mf^2}, \tag{2.12}$$

respectively. Differentiating (2.10) with respect to $\zeta$, the rate of change with height of the radius and the vertical velocity may be expressed in terms of $q$, $f$ and $m$ as

$$\frac{\mathrm{d}\beta}{\mathrm{d}\zeta} = m^{-1/2}\frac{\mathrm{d}q}{\mathrm{d}\zeta} - \frac{1}{2}m^{-3/2}q\frac{\mathrm{d}m}{\mathrm{d}\zeta}, \tag{2.13}$$

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} = \frac{1}{q}\frac{\mathrm{d}m}{\mathrm{d}\zeta} - \frac{m}{q^2}\frac{\mathrm{d}q}{\mathrm{d}\zeta} = \frac{\Gamma_0}{3m}\left(4 - \frac{5}{\Gamma}\right). \tag{2.14}$$

Differentiating (2.12) with respect to $\zeta$ and substituting (2.11) and (2.10) we obtain

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\zeta} = \frac{\Gamma}{\beta}\left(\frac{10}{3}(1-\Gamma) + \Lambda^{1/2}\right), \tag{2.15}$$

$$\frac{\mathrm{d}\Lambda}{\mathrm{d}\zeta} = \frac{\Lambda}{\beta}\left(\frac{10}{3} - \frac{4}{3}\Gamma - 2\Lambda^{1/2}\right). \tag{2.16}$$

Finally, substituting (2.11) and (2.10) into (2.13) yields

$$\frac{\mathrm{d}\beta}{\mathrm{d}\zeta} = \frac{1}{3}(5 - 2\Gamma). \tag{2.17}$$

Thus we have three equations for the three unknowns ($\Gamma$, $\Lambda$, $\beta$)† with initial conditions

$$\Gamma = \Gamma_0, \quad \Lambda = \Lambda_0, \quad \text{and} \quad \beta = 1 \quad \text{at} \quad \zeta = 0. \tag{2.18}$$

From (2.17) and (2.14) it is possible to establish two known results by inspection. For $\Gamma > 5/2$ the plume radius will decrease with height, that is the plume will contract. When $\Gamma = 5/2$ the plume sides will be vertical, referred to as the plume neck. Also, for $\Gamma > 5/4$ the plume velocity will increase with height to a maximum at $\Gamma = 5/4$ (see Caulfield 1991). In the next section we examine steady solutions of these equations and in § 4 we present solutions for the near-source flow when $\Lambda = 0$.

## 3. Steady solutions

We seek steady solutions of (2.15), (2.16) and (2.17) subject to (2.18). For a horizontal source with constant buoyancy flux ($\Lambda = 0$) there are two solutions: the pure-plume solution ($\Gamma = 1$) and the jet solution ($\Gamma = 0$). For $\Lambda > 0$, that is with a vertical buoyancy input, these solutions become unstable, and there is a further solution at ($\Gamma = \frac{5}{4}$, $\Lambda = \frac{25}{36}$) (see Caulfield & Woods 1998 for a more detailed discussion of the stability of these types of flows). Finally, there is a physically unrealistic solution at ($\Gamma = 0$, $\Lambda = \frac{25}{9}$) which is not considered further here. These solutions are clearly visible in the vector plot of $\{\mathrm{d}\Gamma/\mathrm{d}\zeta, \mathrm{d}\Lambda/\mathrm{d}\zeta\}$ in figure 1. We now examine each solution in turn.

† If we had scaled our vertical coordinate $z$ on $L_A$ instead of $L_Q$ then (2.15)–(2.17) would be identical, but for the right-hand side multiplied by $\Gamma_0^{-1/5}$.

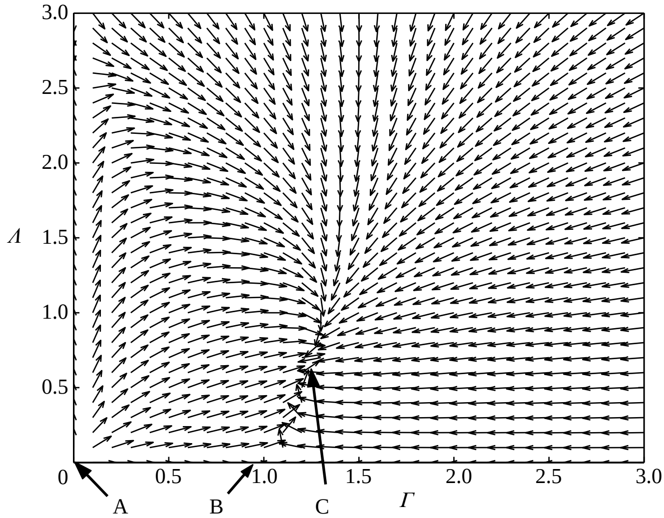


FIGURE 1. Vector plot showing the direction of the vector $\{d\Gamma/d\zeta, d\Lambda/d\zeta\}$ in $(\Gamma, \Lambda)$ space. Note that locally all arrows point away from the jet solution $(0, 0)$ indicating it is unstable to small perturbations in either $\Gamma$ or $\Lambda$. The arrows along $\Gamma = 0$ point to $(1, 0)$, indicating that this solution is attractive provided $\Lambda = 0$. However, for all positive values of $\Lambda$ all paths lead to the stable solution $(\frac{5}{4}, \frac{25}{36})$.

### 3.1. *Jet* $\Gamma = 0$, $\Lambda = 0$

A jet corresponds to both $\Gamma$ and $\Lambda$ being zero. From figure 1 we note that this solution (point A) is unconditionally unstable as locally all arrows point away from $(0, 0)$. That is a jet is only possible if both $\Gamma$ and $\Lambda$ are identically zero for all $z$. Any deviation will result in the flow developing towards one of the other solutions. From (2.17) we see that the radius grows linearly with height as $\beta = \frac{5}{3}\zeta$ or, in dimensional terms, $b = 2\alpha z$.

### 3.2. *Plume with constant buoyancy flux* $\Gamma = 1$, $\Lambda = 0$

For $\Lambda = 0$, any initial value of $\Gamma_0 > 0$ will lead to the pure-plume solution (point B on figure 1) in the far field. However, even an infinitesimally small value of $\Lambda$ will drive the far-field flow towards the $(\frac{5}{4}, \frac{25}{36})$ solution (§ 3.3). Again the radius grows linearly with height as $\beta = \zeta$ or $b = (6\alpha/5)z$. Substituting $\Gamma = 1$, $f = 1$ and $\beta = \zeta$ into (2.10) and (2.12) we obtain the power-law solutions for $Q$ and $M$ without solving the differential equations (2.11) directly:

$$Q = \left(\frac{5F}{4\alpha}\right)^{1/3} \left(\frac{6\alpha z}{5}\right)^{5/3}, \quad M = \left(\frac{5F}{4\alpha}\right)^{2/3} \left(\frac{6\alpha z}{5}\right)^{4/3}. \tag{3.1}$$

Note, however, that this solution does not account for a virtual origin offset, but only evaluates the constant of proportionality in terms of $\alpha$ and the power-law scalings.

### 3.3. *Plume with linear internal buoyancy flux gain* $\Gamma = \frac{5}{4}$, $\Lambda = \frac{25}{36}$

Provided both $\Gamma_0$ and $\Lambda_0$ are greater than zero the flow will tend towards the solution $\Gamma = \frac{5}{4}$, $\Lambda = \frac{25}{36}$ (point C on figure 1). In this case the plume is narrower ($\beta = \frac{5}{6}\zeta$) than a pure plume and lazy ($\Gamma = \frac{5}{4} > 1$). Also, from (2.14), it is a constant-velocity flow. Substituting in the same manner as for the pure-plume case we can solve for the steady-state power-law behaviour of $b$, $F$, $M$ and $Q$ to get

$$b = \alpha z, \quad F \sim \epsilon z, \quad M \sim \epsilon^{2/3} z^2 \sim F^{2/3} z^{4/3}, \quad Q \sim \epsilon^{1/3} z^2 \sim F^{1/3} z^{5/3}. \tag{3.2}$$

Bhat & Narasimha (1996) observed that plumes with off-source heating are narrower than pure plumes as predicted above using a constant-$\alpha$ model. They also observed reduced entrainment. The inference they made was that this implied a reduced entrainment coefficient ($\alpha$). However, as seen in (2.1) the entrainment ($\mathrm{d}Q/\mathrm{d}z$) is related to the product of the entrainment coefficient and the square root of the momentum flux. If the plume is lazy this implies a momentum flux deficit and, therefore, reduced entrainment, even for a constant-$\alpha$ model. Note also that for a fully developed plume with internal buoyancy flux gain the model of Kaminski *et al.* (2005) would predict an entrainment coefficient slightly higher than that for a pure plume.

## 4. Solution for near-source flow with $\Lambda(\zeta) = 0$

We now examine a highly lazy constant-buoyancy-flux plume. However, first we briefly discuss the question of whether or not the MTT model is applicable in the near-source region, as it was derived for slender flows using an assumption of self similarity. Despite this assumption these equations have been successfully applied to flows that are not self-similar such as the near-source flow of a forced plume, Morton (1959), and turbulent fountains, Bloomfield & Kerr (2000). We therefore believe that examination of the MTT conservation equations for the near-source region of a highly lazy plume warrants investigation and may provide valuable insight into the behaviour of these flows.

Equations (2.15), with $\Lambda = 0$, and (2.17), subject to the source conditions $\Gamma = \Gamma_0$ and $\beta = 1$ at $\zeta = 0$, can be integrated to give $\beta$ as a function of $\Gamma$,

$$\beta = \begin{cases} \Gamma^{1/2}(1-\Gamma)^{3/10}\Gamma_0^{-1/2}(1-\Gamma_0)^{-3/10} & \text{for} \quad \Gamma_0 < 1 \\ \Gamma^{1/2}(\Gamma-1)^{-3/10}(\Gamma_0-1)^{3/10}\Gamma_0^{-1/2} & \text{for} \quad \Gamma_0 > 1. \end{cases} \tag{4.1}$$

For $\Gamma_0 > \frac{5}{2}$ the plume will contract immediately above the source (2.17) before beginning to expand further from the source when $\Gamma < \frac{5}{2}$. We can therefore establish from (4.1) a value for the minimum plume radius $\beta_{min}$, namely

$$\beta_{min} = \frac{5^{1/2}}{3^{3/10}2^{1/5}} \frac{(\Gamma_0 - 1)^{3/10}}{\Gamma_0^{1/2}}. \tag{4.2}$$

For $\Gamma_0 \gg 1$ we have $\beta_{min} \approx 1.4\Gamma_0^{-1/5}$. Substituting (4.1) into (2.15) we obtain expressions for the rate of change of $\Gamma$ with height in terms of $\Gamma$ only:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\zeta} = \begin{cases} \Omega\Gamma^{1/2}(1-\Gamma)^{7/10} & \text{for} \quad \Gamma_0 < 1 \\ -\Omega\Gamma^{1/2}(\Gamma-1)^{13/10} & \text{for} \quad \Gamma_0 > 1, \end{cases} \tag{4.3}$$

where $\Omega = \frac{10}{3}\Gamma_0^{1/2}/(\Gamma_0 - 1)^{3/10}$. As expected, (4.3) shows that for lazy plumes $\Gamma$ is a decreasing function of height. From (4.3) we obtain an expression for $\zeta(\Gamma)$:

$$\zeta = \begin{cases} \dfrac{1}{\Omega}\displaystyle\int_{\Gamma_0}^{\Gamma} \Gamma^{-1/2}(\Gamma-1)^{-7/10}\,\mathrm{d}\Gamma & \text{for} \quad \Gamma_0 < 1 \\ -\dfrac{1}{\Omega}\displaystyle\int_{\Gamma_0}^{\Gamma} \Gamma^{-1/2}(\Gamma-1)^{-13/10}\,\mathrm{d}\Gamma & \text{for} \quad \Gamma_0 > 1. \end{cases} \tag{4.4}$$

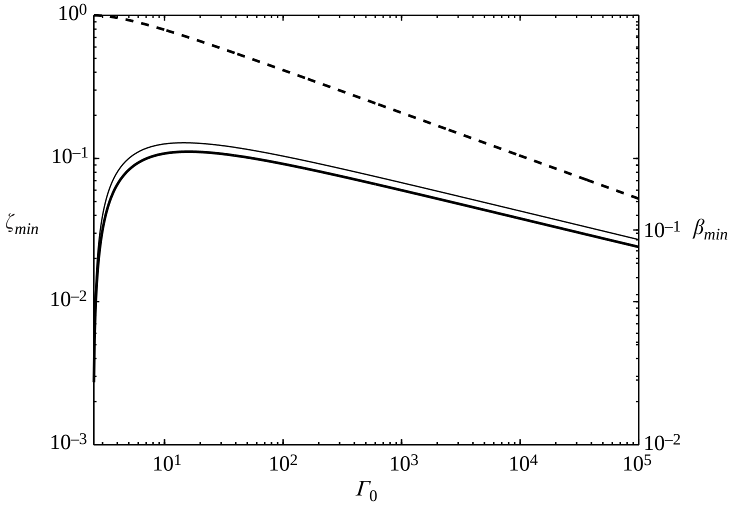


FIGURE 2. The neck height $\zeta_{min}$ and minimum radius $\beta_{min}$ for a lazy plume as a function of the source parameter $\Gamma_0$. The solid lines plotted against the left-hand axis are the values of $\zeta_{min}$ with the thick line representing the numerical solutions and the thin line the analytical approximation (4.7). The dashed line plotted against the right-hand axis is $\beta_{min}$ from (4.2).

The vertical height above the source at which the minimum radius (4.2) is reached is given by (4.4) with the upper limit of integration $\Gamma = \frac{5}{2}$.

For large values of $\Gamma$, that is highly lazy plumes, we make the simplifying approximation that $\Gamma \approx \Gamma - 1$. This reduces (4.3) to

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\zeta} \approx -\Omega(\Gamma - 1)^{9/5}. \tag{4.5}$$

Note that (4.5) gives the correct limit as $\Gamma \to 1$ but will approach that limit more slowly than (4.3). The exact solution of (4.5) is given by

$$\Gamma = 1 + \left(\frac{4}{5}\Omega\zeta + (\Gamma_0 - 1)^{-4/5}\right)^{-5/4}. \tag{4.6}$$

Substituting $\Gamma = \Gamma_{\beta=\beta_{min}} = 5/2$ we get an approximate solution for the height of the neck

$$\zeta_{\beta=\beta_{min}} = \frac{5}{4\Omega}\left(\left(\frac{3}{2}\right)^{-4/5} - (\Gamma_0 - 1)^{-4/5}\right). \tag{4.7}$$

Numerical solutions of (2.15) and (2.17) were determined over the range $2.5 < \Gamma_0 < 10^5$ with $\zeta_{\beta=\beta_{min}}$ calculated and compared to the analytical approximation (4.7). These results (figure 2) show good agreement, though (4.7) consistently overestimates $\zeta_{min}$ as the approximation $\Gamma \approx \Gamma - 1$ tends to underestimate the value of $\mathrm{d}\Gamma/\mathrm{d}\zeta$ resulting in lower $\Gamma$ values at any height. Other than slightly overestimating the neck height, the approximate solution follows well the dependence of $\zeta_{min}$ with $\Gamma_0$ derived from numerical integration.

An approximate solution for the plume radius as a function of height in the near field is established by substituting (4.6) into (4.1) to give

$$\beta \approx 3^{1/4}(3 + 8(\Gamma_0 - 1)\zeta)^{-1/4}. \tag{4.8}$$

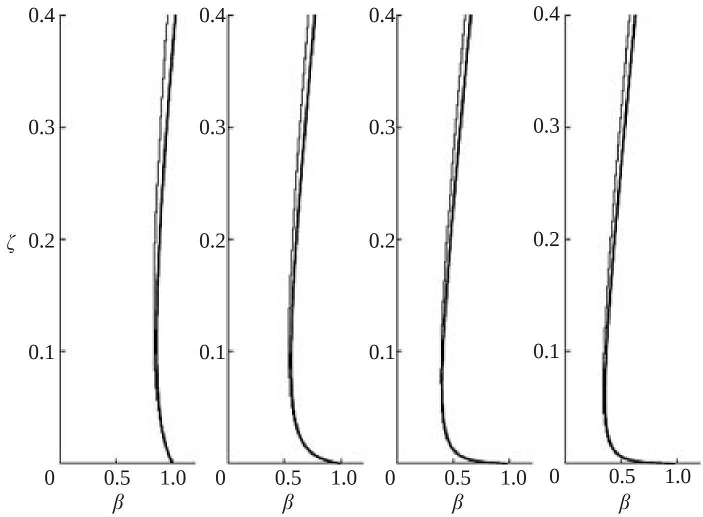


FIGURE 3. Plume radius as a function of height. The thick lines are numerical solutions and the thin lines given by the matched solution (4.9). From left to right $\Gamma_0 = 10$, 100, 500 and 1000.

A first-order approximation for the radius at all heights may be obtained by simply summing the far-field result $\beta = \zeta$ and the near-field result (4.8):

$$\beta \approx \zeta + 3^{1/4}(3 + 8(\Gamma_0 - 1)\zeta)^{-1/4}. \tag{4.9}$$

This approximation is reasonable as the far-field result will have only a small effect on the approximation in the near field where $\zeta$ is small. The approximate solution (4.9) was compared to numerical calculations for a range of source conditions (figure 3). Clearly (4.9) provides a good approximation near the plume source; however, in the far field the agreement is not as good. Although the radial growth rate of the plume is correct, the offset for the virtual origin of the plume is not modelled.

Having established approximate solutions for $\Gamma$ and $\beta$ near the plume source we can now establish the near-source power-law relationships for the plume fluxes $m$ and $q$. These fluxes can be expressed in terms of $\beta$ and $\Gamma$ by rearranging (2.10) and (2.12) to get $m = \beta^{4/3}\Gamma_0^{2/3}\Gamma^{-2/3}$ and $q = \beta^{5/3}\Gamma_0^{1/3}\Gamma^{-1/3}$. This leads to

$$m = \left(\frac{\Gamma_0 - 1}{\Gamma - 1}\right)^{2/5} \approx \left(\frac{8}{3}\Gamma_0\zeta + 1\right)^{1/2}, \tag{4.10}$$

$$q = \left(\frac{\Gamma}{\Gamma_0} \times \frac{1 - \Gamma_0}{1 - \Gamma}\right)^{1/2} = \left(\frac{\Gamma_0 - 1}{\Gamma_0}\right)^{1/2}\left(1 + \frac{1}{2\Gamma} + \cdots\right), \tag{4.11}$$

for $1/\Gamma < 1$. Thus, for $\Gamma \gg 1$, $q \approx 1$ (4.11), i.e. the leading-order term for the volume flux is a constant and, therefore, there is negligible entrainment. For example, in the transition from $\Gamma_0 = 1000$ to $\Gamma = 100$ there is an increase in $q$ of only 0.5%. Again, this result was achieved using a constant entrainment coefficient in our model. In fact, to leading order the flow is independent of $\alpha$. It is therefore reasonable to conclude that the suppressed entrainment observed in lazy plumes (for example Bhat & Narasimha 1996) does not necessarily imply a smaller value of $\alpha$ but rather a need to accurately model the source conditions of the plume.

| Parameter | $\Gamma \approx 0$ | $\Gamma = 1$ | $\Gamma \gg 1$ | $\alpha = 0$ |
|---|---|---|---|---|
| $\beta$ | $\zeta$ | $\zeta$ | $(c+\zeta)^{-1/4}$ | $(c+\zeta)^{-1/4}$ |
| $q$ | $\zeta$ | $\zeta^{5/3}$ | $\zeta^0$ | $\zeta^0$ |
| $m$ | $\zeta^0$ | $\zeta^{4/3}$ | $m_0+\zeta^{1/2}$ | $m_0+\zeta^{1/2}$ |
| $w$ | $\zeta^{-1}$ | $\zeta^{-1/3}$ | $\zeta^{1/2}$ | $\zeta^{1/2}$ |
| $g'$ | $\zeta^{-1}$ | $\zeta^{-5/3}$ | $\zeta^0$ | $\zeta^0$ |
| $\Gamma$ | $\zeta^2$ | $\zeta^0$ | $\zeta^{-5/4}$ | $\zeta^{-5/4}$ |

TABLE 1. Power-law behaviour of various plume parameters with height ($\zeta$) in the near-source region for various values of $\Gamma$. The pure-plume behaviour is as presented by MTT and $c$ is a constant.

Identical scalings to those of (4.10) and (4.11) are obtained on solving the plume conservation equations in the absence of entrainment. On setting $\alpha = 0$, the solution of (2.1) subject to (2.2) yields the constant-volume-flux, constant-buoyancy-flux accelerating flow $q = 1$, $f = 1$ and $M = M_0 + \sqrt{2QFz}$. Based on the results established here and similar analysis for forced plumes we can establish the near-source scalings for all the major plume parameters, as summarized in table 1.

We return now to our original definition of $\Gamma_0$, (2.4), as being the square of the ratio of the momentum jet length ($L_M(0)$) and the source volume flux length scale ($L_Q(0)$). For small $\Gamma_0$, $L_M > L_Q$, the flow behaves like a constant momentum flux jet as there is an excess of momentum flux at the source relative to the equivalent pure plume (Morton 1959). A lazy plume can therefore be regarded as a plume with an excess of volume flux $L_Q > L_M$, see (2.4) and Caulfield & Woods (1995). Therefore, near the source, the flow would be expected to behave as a constant volume flux flow while the momentum flux adjusts to the pure-plume balance ($\Gamma = 1$), analogous to the constant momentum flux adjustment for forced plumes.

## 5. Conclusions

The constant-entrainment-coefficient plume model of MTT has been re-cast as a set of equations for the plume radius $\beta$, flux balance parameter $\Gamma$ and the off-source 'heating' parameter $\Lambda$. Using this set of equations we have shown the far-field flow for a plume with a linear internal buoyancy gain with height to be lazy with a radial growth rate lower than for a pure plume. The solution of these equations also indicates that the velocity of this plume is independent of height. As the plume is lazy in the far field it will have a momentum flux deficit and, therefore, the rate of entrainment will be lower than for a pure plume. This reduced entrainment in plumes with internal buoyancy gains has been cited in the past as implying the need for a variable-entrainment-coefficient model (see Sreenivas & Prasad 2000). However, our analysis implies that the constant-$\alpha$ formulation of MTT also leads to reduced entrainment. We therefore believe that when modelling these flows it is key to establish the correct source conditions and buoyancy flux gain rate before appealing to the additional complexity of a variable-$\alpha$ model.

This formulation was also used to examine the near-source region of highly lazy plumes from horizontal area sources. Approximate solutions were derived for the radius $\beta$ and the flux balance parameter $\Gamma$ in this region. From this, the power-law behaviour for the buoyancy, volume and momentum fluxes and vertical velocity was established. These power-law relationships are summarized in table 1. The constant-$\alpha$

formulation leads to a region of constant volume flux (i.e. zero entrainment) in the near-source region. Again this implies that it is possible to capture reduced entrainment with a constant-$\alpha$ formulation. Though we are not suggesting that $\alpha$ is a constant in all plume flows, it is clear that some reduced entrainment flows can be modelled using a constant $\alpha$.

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